

## Maxwell's equations in materials

### Reading:

- ▶ Kasap: not discussed
- ▶ Griffiths EM: Chapter 7

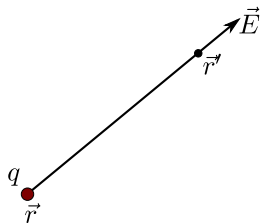
# Electrostatics

- ▶ Coulomb's law: electric field around a point charge  $q$

$$\vec{E}(\vec{r}') = \frac{q(\vec{r}' - \vec{r})}{4\pi\epsilon_0|\vec{r}' - \vec{r}|^3} = -\nabla_{\vec{r}'} \underbrace{\left( \frac{q}{4\pi\epsilon_0|\vec{r}' - \vec{r}|} \right)}_{\phi(\vec{r}')}$$

- ▶ Gauss's law (integral form):

$$\epsilon_0 \int \vec{E} \cdot d\vec{a} = q$$



- ▶ Gauss's law (differential form), for charge density  $\rho(\vec{r})$ :

$$\epsilon_0 \nabla \cdot \vec{E} = \rho \text{ and } \nabla \times \vec{E} = 0$$

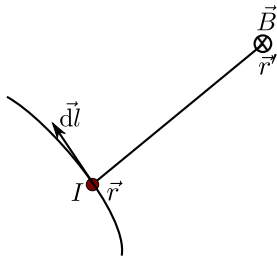
or equivalently, in terms of the potential:

$$-\epsilon_0 \nabla^2 \phi = \rho$$

# Magnetostatics

- ▶ Biot-Savart law: magnetic field around a current  $I$

$$\vec{B}(\vec{r}') = \int \frac{\mu_0 I d\vec{l} \times (\vec{r}' - \vec{r})}{4\pi |\vec{r}' - \vec{r}|^3}$$



- ▶ Ampere's law (integral form)

$$\int \vec{B} \cdot d\vec{l} = \mu_0 I$$

- ▶ Ampere's law (differential form), for current density  $\vec{j}(\vec{r})$ :

$$\frac{1}{\mu_0} \nabla \times \vec{B} = \vec{j} \text{ and } \nabla \cdot \vec{B} = 0$$

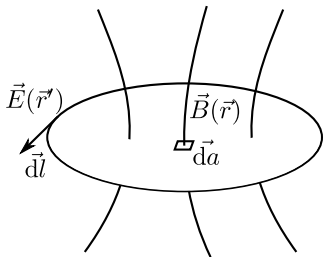
# Electromagnetic induction

- ▶ Faraday's law for electromotive force:

$$\text{EMF} = \oint \vec{dl} \cdot \vec{E}(\vec{r}') = -\frac{d}{dt} \underbrace{\int \vec{da} \cdot \vec{B}(\vec{r})}_{\text{Flux } \Phi}$$

- ▶ Differential form:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$



## Equations so far

$$\epsilon_0 \nabla \cdot \vec{E} = \rho$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\frac{1}{\mu_0} \nabla \times \vec{B} = \vec{j}$$

$$\nabla \cdot \vec{B} = 0$$

Are these correct in general?

Apply divergence to third equation:

$$\nabla \cdot \vec{j} = \frac{1}{\mu_0} \nabla \cdot (\nabla \times \vec{B}) = 0$$

Divergence of current is the rate at which charge leaves a point (continuity equation i.e. charge conservation):

$$\begin{aligned} \nabla \cdot \vec{j} &= -\frac{\partial \rho}{\partial t} \\ &= -\frac{\partial}{\partial t} (\epsilon_0 \nabla \cdot \vec{E}) \end{aligned}$$

So how can we fix the equations?

# Maxwell's equations

$$\epsilon_0 \nabla \cdot \vec{E} = \rho$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\frac{1}{\mu_0} \nabla \times \vec{B} = \vec{j} + \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\nabla \cdot \vec{B} = 0$$

Apply divergence to third equation:

$$\nabla \cdot \vec{j} + \frac{\partial(\epsilon_0 \nabla \cdot \vec{E})}{\partial t} = \frac{1}{\mu_0} \nabla \cdot (\nabla \times \vec{B}) = 0$$

$$\Rightarrow \nabla \cdot \vec{j} + \frac{\partial \rho}{\partial t} = 0$$

Now consistent with charge conservation.

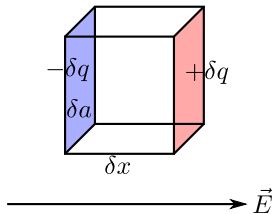
# Materials in electric fields

- ▶ All materials composed of charges: electrons and nuclei
- ▶ Charges pulled along/opposite electric field with force  $q\vec{E}$
- ▶ Charges separated in each infinitesimal chunk of matter  $\Rightarrow$  dipoles
- ▶ Induced dipole moment:

$$\delta\vec{p} = \delta q \delta x \hat{x}$$

- ▶ Polarization is the density of induced dipoles:

$$\vec{P} = \frac{\delta\vec{p}}{\delta x \delta a} = \frac{\delta q}{\delta a} \hat{x}$$



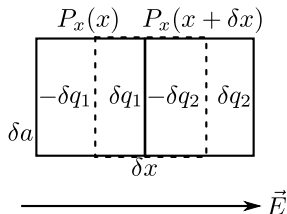
## Bound charge due to polarization

- ▶ Charge density in infinitesimal chunk

$$\begin{aligned}
 \rho_b &= \frac{\delta q_1 - \delta q_2}{\delta a \delta x} \\
 &= \frac{\frac{\delta q_1}{\delta a} - \frac{\delta q_2}{\delta a}}{\delta x} \\
 &= \frac{P_x(x) - P_x(x + \delta x)}{\delta x} \\
 &= -\frac{\partial P_x}{\partial x}
 \end{aligned}$$

- ▶ Similarly accounting for  $y$  and  $z$  components:

$$\rho_b = -\nabla \cdot \vec{P}$$





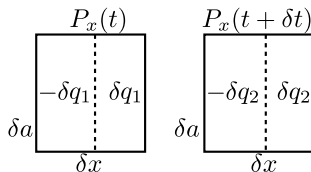
## Current density due to polarization

- ▶ Charge crossing dotted surface in time  $\delta t$ :

$$\delta q = \delta q_2 - \delta q_1$$

- ▶ Corresponding current density:

$$\begin{aligned} j_x &= \frac{\delta q_2 - \delta q_1}{\delta a \delta t} \\ &= \frac{\frac{\delta q_2}{\delta a} - \frac{\delta q_1}{\delta a}}{\delta t} \\ &= \frac{P_x(t + \delta t) - P_x(t)}{\delta t} \\ &= \frac{\partial P_x}{\partial t} \end{aligned}$$



- ▶ Polarization current density in general direction:

$$\vec{j}_P = \frac{\partial \vec{P}}{\partial t}$$

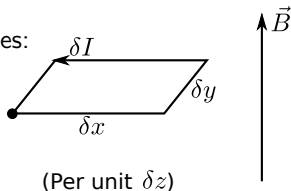
## Materials in magnetic fields

- ▶ Charges circulate around magnetic field due to force  $q\vec{v} \times \vec{B}$
- ▶ Magnetic dipole moment of infinitesimal current loop (per unit  $\delta z$ )

$$\begin{aligned}\vec{\delta\mu} &= \frac{1}{2} \oint \vec{r} \times d\vec{l} \delta I \\ &= \frac{1}{2} (0 + \delta x \delta y \hat{z} \delta I + \delta y \delta x \hat{z} \delta I + 0) \\ &= \delta x \delta y \hat{z} \delta I\end{aligned}$$

- ▶ Magnetization is density of induced magnetic dipoles:

$$\vec{M} = \frac{\delta\mu}{\delta x \delta y} = \hat{z} \delta I$$



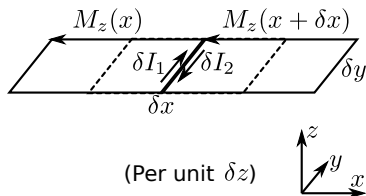
## Bound current density due to magnetization

- ▶ Current within dotted element (per unit  $\delta z$ )

$$\delta I_y = \delta I_1 - \delta I_2$$

- ▶ Corresponding current density:

$$\begin{aligned} j_y &= \frac{I_y}{\delta x} = \frac{\delta I_1 - \delta I_2}{\delta x} \\ &= \frac{M_z(x) - M_z(x + \delta x)}{\delta x} \\ &= -\frac{\partial M_z}{\partial x} \end{aligned}$$



- ▶ Generalizing to all directions:

$$\vec{j}_b = \nabla \times \vec{M}$$

## Material response summary

- ▶ Response to electric field  $\vec{E}$  is polarization  $\vec{P}$
- ▶ Polarization corresponds to bound charge density

$$\rho_b = -\nabla \cdot \vec{P}$$

and current density

$$\vec{j}_P = \frac{\partial \vec{P}}{\partial t}$$

- ▶ Response to magnetic field  $\vec{B}$  is magnetization  $\vec{M}$
- ▶ Magnetization corresponds to bound current density

$$\vec{j}_b = \nabla \times \vec{M}$$

## Maxwell's equations including material response

- ▶ Fields produced by external 'free' charges and currents ( $\rho_f$  and  $\vec{j}_f$ ) as well as bound ones induced in the materials

$$\begin{aligned} \epsilon_0 \nabla \cdot \vec{E} &= (\rho_f + \rho_b) & \frac{1}{\mu_0} \nabla \times \vec{B} &= (\vec{j}_f + \vec{j}_b + \vec{j}_P) + \epsilon_0 \frac{\partial \vec{E}}{\partial t} \\ \nabla \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} & \nabla \cdot \vec{B} &= 0 \end{aligned}$$

- ▶ Rewrite bound quantities in terms of polarization and magnetization

$$\begin{aligned} \epsilon_0 \nabla \cdot \vec{E} &= (\rho_f + \rho_b) & \frac{1}{\mu_0} \nabla \times \vec{B} &= (\vec{j}_f + \vec{j}_b + \vec{j}_P) + \epsilon_0 \frac{\partial \vec{E}}{\partial t} \\ \epsilon_0 \nabla \cdot \vec{E} &= \rho_f - \nabla \cdot \vec{P} & \frac{1}{\mu_0} \nabla \times \vec{B} &= \vec{j}_f + \nabla \times \vec{M} + \frac{\partial \vec{P}}{\partial t} + \epsilon_0 \frac{\partial \vec{E}}{\partial t} \\ \nabla \cdot (\epsilon_0 \vec{E} + \vec{P}) &= \rho_f & \nabla \times \left( \frac{\vec{B}}{\mu_0} - \vec{M} \right) &= \vec{j}_f + \frac{\partial (\epsilon_0 \vec{E} + \vec{P})}{\partial t} \end{aligned}$$

# Maxwell's equations in media

- ▶ Define fields

$$\vec{D} \equiv \epsilon_0 \vec{E} + \vec{P}$$

$$\vec{H} \equiv \frac{\vec{B}}{\mu_0} - \vec{M}$$

- ▶ Yields equations with free charge and current densities as the sources:

$$\nabla \cdot \vec{D} = \rho_f$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{H} = \vec{j}_f + \frac{\partial \vec{D}}{\partial t}$$

$$\nabla \cdot \vec{B} = 0$$

## Constitutive relations

- ▶ Material determines how  $\vec{P}$  (and hence  $\vec{D}$ ) depends on  $\vec{E}$
- ▶ Material determines how  $\vec{M}$  (and hence  $\vec{H}$ ) depends on  $\vec{B}$
- ▶ Simplest case: linear isotropic dielectric

$$\vec{P} = \chi_e \epsilon_0 \vec{E}$$

$$\vec{M} = \chi_m \vec{H}$$

$$\vec{D} = (1 + \chi_e) \epsilon_0 \vec{E}$$

$$\vec{B} = (1 + \chi_m) \mu_0 \vec{H}$$

$$\epsilon = (1 + \chi_e) \epsilon_0$$

$$\mu = (1 + \chi_m) \mu_0$$

- ▶ Anisotropic dielectric:  $\vec{P} = \bar{\chi}_e \cdot \epsilon_0 \vec{E}$  with susceptibility tensor  $\bar{\chi}_e$
- ▶ Nonlinear dielectric:  $\vec{P} = \chi_e(E) \epsilon_0 \vec{E}$

# Ohm's law

- ▶ Response of metals to constant electric fields given by

$$\vec{j} = \sigma \vec{E}$$

with electrical conductivity  $\sigma$

- ▶ But what is the corresponding constitutive relation  $\vec{P}(\vec{E})$ ?
- ▶ The current is actually a polarization current  $\vec{j}_P = \sigma \vec{E}$
- ▶ Remember  $\vec{j}_P = \partial \vec{P} / \partial t$ , so

$$\vec{P} = \int dt \vec{j}_P = \vec{P}_0 + t\sigma \vec{E}$$

- ▶ Important: the response is not instantaneous in general
- ▶ It can depend on the history i.e. is non-local in time



## Frequency domain

- ▶ For linear materials, convenient to work in frequency domain where all quantities have time dependence  $f(t) \equiv f e^{-i\omega t}$  with angular frequency  $\omega$
- ▶ Maxwell's equations take the form

$$\nabla \cdot \vec{D} = \rho_f$$

$$\nabla \times \vec{E} = i\omega \vec{B}$$

$$\nabla \times \vec{H} = \vec{j}_f - i\omega \vec{D}$$

$$\nabla \cdot \vec{B} = 0$$

- ▶ For Ohmic metal (with frequency-dependent conductivity  $\sigma(\omega)$ ):

$$\sigma(\omega) \vec{E} = \vec{j}_P \equiv \partial \vec{P} / \partial t = -i\omega \vec{P}$$

$$\Rightarrow \vec{P} = \frac{i\sigma(\omega) \vec{E}}{\omega}$$

$$\Rightarrow \epsilon(\omega) = \epsilon_0 + \frac{i\sigma(\omega)}{\omega}$$

## Electromagnetic waves

- ▶ Linear response of materials described very generally by  $\epsilon(\omega)$  and  $\mu(\omega)$
- ▶ Maxwell's equations in the absence of free charges and currents

$$\nabla \cdot (\epsilon(\omega)\vec{E}) = 0$$

$$\nabla \times \vec{E} = i\omega\vec{B}$$

$$\nabla \times \frac{\vec{B}}{\mu(\omega)} = -i\omega(\epsilon(\omega)\vec{E})$$

$$\nabla \cdot \vec{B} = 0$$

- ▶ Substitute second equation in curl of third equation:

$$\frac{\nabla \times (\nabla \times \vec{B})}{\mu(\omega)} = -i\omega\epsilon(\omega)\nabla \times \vec{E} = \omega^2\epsilon(\omega)\vec{B}$$

$$-\nabla^2\vec{B} = \omega^2\epsilon(\omega)\mu(\omega)\vec{B}$$

$$\text{using } \nabla \cdot \vec{B} = 0$$

## Electromagnetic wave equation

- ▶ Write Maxwell's equation in linear media with no free charge or current as:

$$v^2(\omega)\nabla^2\vec{B} = -\omega^2\vec{B}$$

$$v^2(\omega)\nabla^2\vec{E} = -\omega^2\vec{E}$$

where  $v(\omega) \equiv 1/\sqrt{\epsilon(\omega)\mu(\omega)}$

- ▶ First consider 1D version,  $v^2\frac{\partial^2 f}{\partial x^2} = -\omega^2 f$ . General solution:

$$f(x) = A \exp(ikx) + B \exp(-ikx)$$

with  $k = \omega/v$

- ▶ General solution for  $v^2\nabla^2 f = -\omega^2 f$ :

$$f(\vec{r}) = \sum_{\vec{k}} A_{\vec{k}} \exp(i\vec{k} \cdot \vec{r})$$

with  $|\vec{k}| = \omega/v$

## Electromagnetic wave speed

- ▶ Going back to EM waves and to time domain:

$$\vec{E}(\vec{r}, t) = \sum_{\vec{k}} \vec{E}_{\vec{k}0} \exp(i\vec{k} \cdot \vec{r} - i\omega t)$$

$$\vec{B}(\vec{r}, t) = \sum_{\vec{k}} \vec{B}_{\vec{k}0} \exp(i\vec{k} \cdot \vec{r} - i\omega t)$$

with  $E_0 \perp B_0 \perp \vec{k}$  to satisfy Maxwell's equations

- ▶ For each  $\vec{k}$ , two linearly-independent choices for  $\vec{E}$  (polarizations)
- ▶ Wave speed is  $v(\omega) = \omega/|\vec{k}| = 1/\sqrt{\epsilon(\omega)\mu(\omega)}$
- ▶ In vacuum,  $\epsilon(\omega) = \epsilon_0$  and  $\mu(\omega) = \mu_0$ , so speed of light  $c = 1/\sqrt{\epsilon_0\mu_0}$
- ▶ In materials, speed of light usually specified by refractive index

$$n(\omega) \equiv \frac{c}{v(\omega)} = \sqrt{\frac{\epsilon(\omega)\mu(\omega)}{\epsilon_0\mu_0}}$$