MTLE-6120: Advanced Electronic Properties of Materials

Maxwell's equations in materials

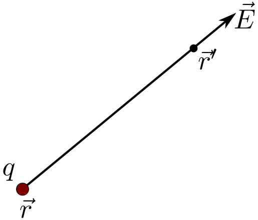
Reading:

- Kasap: not discussed
- Griffiths EM: Chapter 7

Electrostatics

- Coulomb's law: electric field around a point charge q

$$\vec{E}(\vec{r}') = \frac{q(\vec{r}' - \vec{r})}{4\pi\epsilon_0|\vec{r}' - \vec{r}|^3} = -\nabla_{\vec{r}'}\Bigg(\underbrace{\frac{q}{4\pi\epsilon_0|\vec{r}' - \vec{r}|}}_{\phi(\vec{r}')}\Bigg)$$



- Gauss's law (integral form):

$$\epsilon_0 \int \vec{E} \cdot \mathrm{d}\vec{a} = q$$

- Gauss's law (differential form), for charge density $\rho(\vec{r})$:

$$\epsilon_0 \nabla \cdot \vec{E} = \rho \text{ and } \nabla \times \vec{E} = 0$$

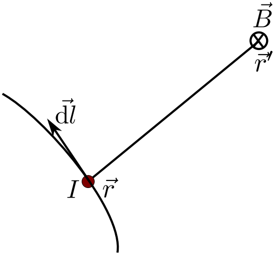
or equivalently, in terms of the potential:

$$-\epsilon_0 \nabla^2 \phi = \rho$$

Magnetostatics

- Biot-Savart law: magnetic field around a current I

$$\vec{B}(\vec{r}') = \int \frac{\mu_0 I \vec{\mathrm{d}l} \times (\vec{r}' - \vec{r})}{4\pi|\vec{r}' - \vec{r}|^3}$$



- Ampere's law (integral form)

$$\int \vec{B} \cdot \mathrm{d}\vec{l} = \mu_0 I$$

- Ampere's law (differential form), for current density $\vec{j}(\vec{r})$:

$$\frac{1}{\mu_0} \nabla \times \vec{B} = \vec{j} \text{ and } \nabla \cdot \vec{B} = 0$$

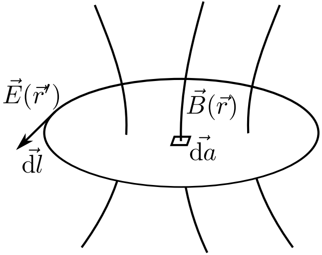
Electromagnetic induction

- Faraday's law for electromotive force:

$$\mathrm{EMF} = \oint \mathrm{d}\vec{l} \cdot \vec{E}(\vec{r}') = -\frac{\mathrm{d}}{\mathrm{d}t} \underbrace{\int \mathrm{d}\vec{a} \cdot \vec{B}(\vec{r})}_{\text{Flux } \Phi}$$

- Differential form:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$



Equations so far

$$\epsilon_0 \nabla \cdot \vec{E} = \rho$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\frac{1}{\mu_0} \nabla \times \vec{B} = \vec{j}$$

$$\nabla \cdot \vec{B} = 0$$

Are these correct in general?

Apply divergence to third equation:

$$\nabla \cdot \vec{j} = \frac{1}{\mu_0} \nabla \cdot \left(\nabla \times \vec{B} \right) = 0$$

Divergence of current is the rate at which charge leaves a point (continuity equation i.e. charge conservation):

$$\begin{aligned} \nabla \cdot \vec{j} &= -\frac{\partial \rho}{\partial t} \\ &= -\frac{\partial}{\partial t} \left(\epsilon_0 \nabla \cdot \vec{E} \right) \end{aligned}$$

So how can we fix the equations?

Maxwell's equations

$$\epsilon_0 \nabla \cdot \vec{E} = \rho$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\frac{1}{\mu_0} \nabla \times \vec{B} = \vec{j} + \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\nabla \cdot \vec{B} = 0$$

Apply divergence to third equation:

$$\nabla \cdot \vec{j} + \frac{\partial(\epsilon_0 \nabla \cdot \vec{E})}{\partial t} = \frac{1}{\mu_0} \nabla \cdot \left(\nabla \times \vec{B} \right) = 0$$

$$\Rightarrow \nabla \cdot \vec{j} + \frac{\partial \rho}{\partial t} = 0$$

Now consistent with charge conservation.

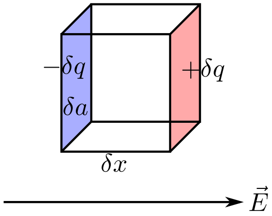
Materials in electric fields

- All materials composed of charges: electrons and nuclei
- Charges pulled along/opposite electric field with force $q\vec{E}$
- Charges separated in each infinitesimal chunk of matter $\Rightarrow$ dipoles
- Induced dipole moment:

$$\delta\vec{p} = \delta q \delta x \hat{x}$$

- Polarization is the density of induced dipoles:

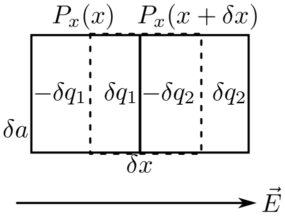
$$\vec{P} = \frac{\delta\vec{p}}{\delta x \delta a} = \frac{\delta q}{\delta a}\hat{x}$$



Bound charge due to polarization

- Charge density in infinitesimal chunk

$$\begin{aligned}\rho_b &= \frac{\delta q_1 - \delta q_2}{\delta a \delta x} \\ &= \frac{\frac{\delta q_1}{\delta a} - \frac{\delta q_2}{\delta a}}{\delta x} \\ &= \frac{P_x(x) - P_x(x+\delta x)}{\delta x} \\ &= -\frac{\partial P_x}{\partial x}\end{aligned}$$



- Similarly accounting for y and z components:

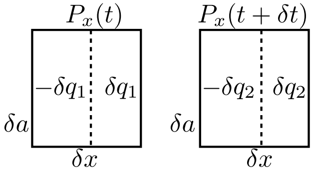
$$\rho_b = -\nabla \cdot \vec{P}$$

Current density due to polarization

- Charge crossing dotted surface in time δt:

$$\delta q = \delta q_2 - \delta q_1$$

- Corresponding current density:



$$\begin{aligned} j_x &= \frac{\delta q_2 - \delta q_1}{\delta a \delta t} \\ &= \frac{\frac{\delta q_2}{\delta a} - \frac{\delta q_1}{\delta a}}{\delta t} \\ &= \frac{P_x(t+\delta t) - P_x(t)}{\delta t} \\ &= \frac{\partial P_x}{\partial t} \end{aligned}$$

- Polarization current density in general direction:

$$\vec{j}_P = \frac{\partial \vec{P}}{\partial t}$$

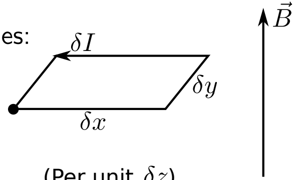
Materials in magnetic fields

- Charges circulate around magnetic field due to force $q\vec{v} \times \vec{B}$
- Magnetic dipole moment of infinitesimal current loop (per unit δz)

$$\begin{aligned}
\delta\vec{\mu} &= \frac{1}{2}\oint \vec{r} \times \mathrm{d}\vec{l}\delta I \\
&= \frac{1}{2}\left(0 + \delta x \delta y \hat{z} \delta I + \delta y \delta x \hat{z} \delta I + 0\right) \\
&= \delta x \delta y \hat{z} \delta I
\end{aligned}$$

- Magnetization is density of induced magnetic dipoles:

$$\vec{M} = \frac{\delta \mu}{\delta x \delta y} = \hat{z}\delta I$$



(Per unit δz)

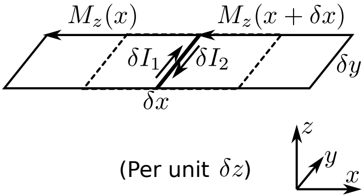
Bound current density due to magnetization

- Current within dotted element (per unit δz)

$$\delta I_y = \delta I_1 - \delta I_2$$

- Corresponding current density:

$$\begin{aligned} j_y = \frac{I_y}{\delta x} = \frac{\delta I_1 - \delta I_2}{\delta x} \\ = \frac{M_z(x) - M_z(x+\delta x)}{\delta x} \\ = -\frac{\partial M_z}{\partial x} \end{aligned}$$



- Generalizing to all directions:

$$\vec{j}_b = \nabla \times \vec{M}$$

Material response summary

- Response to electric field $\vec{E}$ is polarization $\vec{P}$
- Polarization corresponds to bound charge density

$$\rho_b = -\nabla \cdot \vec{P}$$

and current density

$$\vec{j}_P = \frac{\partial \vec{P}}{\partial t}$$

- Response to magnetic field $\vec{B}$ is magnetization $\vec{M}$
- Magnetization corresponds to bound current density

$$\vec{j}_b = \nabla \times \vec{M}$$

Maxwell's equations including material response

- Fields produced by external 'free' charges and currents (ρ_f and $\vec{j}_f$) as well as bound ones induced in the materials

$$\epsilon_0 \nabla \cdot \vec{E} = (\rho_f + \rho_b) \qquad \frac{1}{\mu_0}\nabla \times \vec{B} = (\vec{j}_f + \vec{j}_b + \vec{j}_P) + \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \nabla \cdot \vec{B} = 0$$

- Rewrite bound quantities in terms of polarization and magnetization

$$\epsilon_0 \nabla \cdot \vec{E} = (\rho_f + \rho_b) \qquad \frac{1}{\mu_0}\nabla \times \vec{B} = (\vec{j}_f + \vec{j}_b + \vec{j}_P) + \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\epsilon_0 \nabla \cdot \vec{E} = \rho_f - \nabla \cdot \vec{P} \qquad \frac{1}{\mu_0}\nabla \times \vec{B} = \vec{j}_f + \nabla \times \vec{M} + \frac{\partial \vec{P}}{\partial t} + \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\nabla \cdot (\epsilon_0 \vec{E} + \vec{P}) = \rho_f \qquad \nabla \times \left(\frac{\vec{B}}{\mu_0} - \vec{M}\right) = \vec{j}_f + \frac{\partial(\epsilon_0 \vec{E} + \vec{P})}{\partial t}$$

Maxwell's equations in media

- Define fields

$$\vec{D} \equiv \epsilon_0 \vec{E} + \vec{P}$$
$$\vec{H} \equiv \frac{\vec{B}}{\mu_0} - \vec{M}$$

- Yields equations with free charge and current densities as the sources:

$$\nabla \cdot \vec{D} = \rho_f$$
$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$
$$\nabla \times \vec{H} = \vec{j}_f + \frac{\partial \vec{D}}{\partial t}$$
$$\nabla \cdot \vec{B} = 0$$

Constitutive relations

- Material determines how $\vec{P}$ (and hence $\vec{D}$) depends on $\vec{E}$
- Material determines how $\vec{M}$ (and hence $\vec{H}$) depends on $\vec{B}$
- Simplest case: linear isotropic dielectric

$$\begin{aligned} \vec{P} &= \chi_e \epsilon_0 \vec{E} \\ \vec{D} &= (1+\chi_e)\epsilon_0 \vec{E} \\ \epsilon &= (1+\chi_e)\epsilon_0 \end{aligned} \qquad \begin{aligned} \vec{M} &= \chi_m \vec{H} \\ \vec{B} &= (1+\chi_m)\mu_0 \vec{H} \\ \mu &= (1+\chi_m)\mu_0 \end{aligned}$$

- Anisotropic dielectric: $\vec{P} = \bar{\chi_e} \cdot \epsilon_0 \vec{E}$ with susceptibility tensor $\bar{\chi_e}$
- Nonlinear dielectric: $\vec{P} = \chi_e(E)\epsilon_0 \vec{E}$

Ohm's law

- Response of metals to constant electric fields given by

$$\vec{j} = \sigma \vec{E}$$

with electrical conductivity σ

- But what is the corresponding constitutive relation $\vec{P}(\vec{E})$?
- The current is actually a polarization current $\vec{j}_P = \sigma \vec{E}$
- Remember $\vec{j}_P = \partial \vec{P} / \partial t$, so

$$\vec{P} = \int \mathrm{d}t \vec{j}_P = \vec{P}_0 + t\sigma \vec{E}$$

- Important: the response is not instantaneous in general
- It can depend on the history i.e. is non-local in time

Frequency domain

- For linear materials, convenient to work in frequency domain where all quantities have time dependence $f(t) \equiv f e^{-i\omega t}$ with angular frequency ω
- Maxwell's equations take the form

$$\begin{aligned}
\nabla \cdot \vec{D} &= \rho_f \\
\nabla \times \vec{E} &= i\omega \vec{B} \\
\nabla \times \vec{H} &= \vec{j}_f - i\omega \vec{D} \\
\nabla \cdot \vec{B} &= 0
\end{aligned}$$

- For Ohmic metal (with frequency-dependent conductivity $\sigma(\omega)$):

$$\begin{aligned}
\sigma(\omega)\vec{E} = \vec{j}_P \equiv \partial \vec{P}/\partial t &= -i\omega \vec{P} \\
\Rightarrow \vec{P} &= \frac{i\sigma(\omega)\vec{E}}{\omega} \\
\Rightarrow \epsilon(\omega) &= \epsilon_0 + \frac{i\sigma(\omega)}{\omega}
\end{aligned}$$

Electromagnetic waves

- Linear response of materials described very generally by $\epsilon(\omega)$ and $\mu(\omega)$
- Maxwell's equations in the absence of free charges and currents

$$\begin{aligned}
\nabla \cdot (\epsilon(\omega)\vec{E}) &= 0 \\
\nabla \times \vec{E} &= i\omega\vec{B} \\
\nabla \times \frac{\vec{B}}{\mu(\omega)} &= -i\omega(\epsilon(\omega)\vec{E}) \\
\nabla \cdot \vec{B} &= 0
\end{aligned}$$

- Substitute second equation in curl of third equation:

$$\begin{aligned}
\frac{\nabla \times (\nabla \times \vec{B})}{\mu(\omega)} &= -i\omega\epsilon(\omega)\nabla \times \vec{E} = \omega^2\epsilon(\omega)\vec{B} \\
-\nabla^2\vec{B} &= \omega^2\epsilon(\omega)\mu(\omega)\vec{B} && \text{using } \nabla \cdot \vec{B} = 0
\end{aligned}$$

Electromagnetic wave equation

- Write Maxwell's equation in linear media with no free charge or current as:

$$v^2(\omega)\nabla^2\vec{B} = -\omega^2\vec{B}$$
$$v^2(\omega)\nabla^2\vec{E} = -\omega^2\vec{E}$$

where $v(\omega) \equiv 1/\sqrt{\epsilon(\omega)\mu(\omega)}$

- First consider 1D version, $v^2\frac{\partial^2 f}{\partial x^2} = -\omega^2 f$. General solution:

$$f(x) = A\exp(ikx) + B\exp(-ikx)$$

with $k = \omega/v$

- General solution for $v^2\nabla^2 f = -\omega^2 f$:

$$f(\vec{r}) = \sum_{\vec{k}} A_{\vec{k}}\exp(i\vec{k}\cdot\vec{r})$$

with $|\vec{k}| = \omega/v$

Electromagnetic wave speed

- Going back to EM waves and to time domain:

$$\vec{E}(\vec{r},t) = \sum_{\vec{k}} \vec{E}_{\vec{k}0} \exp(i\vec{k}\cdot\vec{r} - i\omega t)$$

$$\vec{B}(\vec{r},t) = \sum_{\vec{k}} \vec{B}_{\vec{k}0} \exp(i\vec{k}\cdot\vec{r} - i\omega t)$$

  with $E_0 \perp B_0 \perp \vec{k}$ to satisfy Maxwell's equations
- For each $\vec{k}$, two linearly-independent choices for $\vec{E}$ (polarizations)
- Wave speed is $v(\omega) = \omega/|\vec{k}| = 1/\sqrt{\epsilon(\omega)\mu(\omega)}$
- In vacuum, $\epsilon(\omega) = \epsilon_0$ and $\mu(\omega) = \mu_0$, so speed of light $c = 1/\sqrt{\epsilon_0\mu_0}$
- In materials, speed of light usually specified by refractive index

$$n(\omega) \equiv \frac{c}{v(\omega)} = \sqrt{\frac{\epsilon(\omega)\mu(\omega)}{\epsilon_0\mu_0}}$$