# HW9 solution

MTLE-6120: Spring 2023

Due: March 31, 2023

1. **Kasap 8.17: Superconductivity and critical current density**

   (a) The current in Sn when the surface field reaches the critical value is $I_{\max} = 2\pi r B/\mu_0 = 2\pi(5 \times 10^{-4})(0.2)/(4\pi \times 10^{-7}) = 500$ A. The corresponding current density is $j_{\max} = (500\text{ A})/(\pi(5 \times 10^{-4}\text{ m})^2) = 6.4 \times 10^8\text{ A/m}^2$.

   (b) Correspondingly for $B_{c2} = 24.5$ T in $Nb_3Sn$, $I_{\max} = 6.1 \times 10^4$ A and $j_{\max} = 7.810\text{ A/m}^2$.

   This maximum current density is close to the critical $j_c = 10^{11}\text{ A/m}^2$, but that is not always true. Remember that the $I_{\max}$ and $j_{\max}$ depend on the wire diameter, whereas $j_c$ is a geometry-independent material property.

2. **Critical magnetic field of a superconductor**

   Consider a metal with some density of states per unit volume $g(E)$ in its normal state, which becomes a BCS superconductor below a critical temperature $T_c$. In BCS theory, the superconducting gap at zero temperature is given by $\Delta = 1.76 k_B T_c$.

   (a) If all the electrons with energy $E_F - \Delta < E < E_F$ pair up with binding energy $\Delta$ per pair, what is the gain in energy density of the superconductor relative to the normal metal?

   The number of electrons per unit volume in that energy range is approximately $g(E_F)\Delta$, so the number of pairs formed per unit volume is $g(E_F)\Delta/2$, each with energy gain $\Delta$ Therefore, the energy density gained relative to the normal metal is $g(E_F)\Delta^2/2$.

   (b) What is the energy density incurred in expelling a magnetic field $B$ due to the Meissner effect?

   The supercodnuctor must produce an equal and opposite magnetic field $B$ to cancel the external field. The energy density of this magnetic field is $B^2/(2\mu_0)$.

   (c) Given that at the critical magnetic field $B_c$, it is no longer energetically favorable to expel the magnetic field, relate $B_c$ to $\Delta$ (at $T = 0$).

   The energy cost of expelling the magnetic field is equal to the energy gain of the superconducting state at $B = B_c$, so $B_c^2/(2\mu_0) = g(E_F)\Delta^2/2 \Rightarrow B_c = \sqrt{\mu_0 g(E_F)}\Delta$.

   (d) Aluminum is face-centered cubic metal with a cubic lattice constant of 4.05 Å, which behaves almost perfectly like a free-electron metal with 3 free electrons per atom. What is its $g(E_F)$ in SI units ($J^{-1}m^{-3}$)?

   The density of states in a free electron metal is

$$g(E) = \frac{\sqrt{E}}{2\pi^2}\left(\frac{\sqrt{2m}}{\hbar}\right)^3$$

   and the Fermi energy is

$$E_F = \frac{\hbar^2}{2m}(3\pi^2 n)^{2/3}$$

   where $n$ is the number density of free electrons. Putting these together,

$$g(E_F) = \frac{m(3\pi^2 n)^{1/3}}{\pi^2\hbar^2}.$$

For aluminum, the unit cell volume is $\Omega = (4.05\ \text{Å})^3/4 = 1.66 \times 10^{-29}\ \text{m}^3$ and the number density is $n = 3/\Omega = 1.81 \times 10^{29}$ m$^{-3}$. Substituting above, $g(E_F) = 1.43 \times 10^{47}$ J$^{-1}$m$^{-3}$.

(e) Given that aluminum becomes a BCS superconductor below $T_c = 1.2$ K, estimate its zero-temperature critical magnetic field $B_c$ in SI units (Tesla).

$$\begin{aligned} B_c &= \sqrt{\mu_0 g(E_F)}\Delta = \sqrt{\mu_0 g(E_F)} \cdot 1.76 k_B T_c \\ &= \sqrt{4\pi \times 10^{-7} \cdot 1.43 \times 10^{47}} \cdot 1.76 \cdot 1.38 \times 10^{-23} \cdot 1.2\ \text{T} \\ &\approx 0.012\ \text{T} \end{aligned}$$