HW8 solution

MTLE-6120: Spring 2018

Due: Apr 9, 2018

1. **Kasap 8.6: Ferromagnetism and the exchange interaction**

   Note that there are effectively three parts to this question, as listed below for clarity:

   (a) What is the spin magnetic moment of the isolated Dy atom based on its electronic configuration?

   The spin magnetic moment is due to $4f^{10}$, which given that there are seven f orbitals, corresponds to $\uparrow\downarrow, \uparrow\downarrow, \uparrow\downarrow, \uparrow, \uparrow, \uparrow, \uparrow$, and therefore four unpaired spins. The corresponding magnetic moment is $4 \times g\mu_B/2 \approx 4\mu_B$.

   (b) What is the magnetic moment per Dy atom in the solid based on the saturation magnetization? Compare with (a).

   From the density and atomic mass, the number density of atoms is (8.54 g/cm^3) / (162.50 g/mol) $\times\ 6.022 \times 10^{23}$/mol $= 3.16 \times 10^{22}$/cm^3. The saturation magnetization per unit volume is 2.4×10^6 A/m, which corresponds to $(2.4 \times 10^6$ A/m$)/(3.16 \times 10^{22}$/cm$^3) = 7.6 \times 10^{-23}$ Am^2 per atom. Since the Bohr magneton is 9.274×10^{-24} Am^2, this magnetization corresponds to 8.2 μ_B or 8.2 spins per atom.

   Therefore the magnetization per atom in the solid is twice that in the isolated atom. This is perfectly reasonable because in metals, the number of unpaired spins depends on where the Fermi level is relative to the band density of states. This can differ considerably form the atomic picture.

   (c) Estimate the exchange interaction magnitude given the Curie temperature.

   Assuming $E_{ex} \sim k_B T_c$ with $T_c \sim 85$ K yields $E_{ex} \sim 7.4 \times 10^{-3}$ eV.

2. **Kasap 8.17: Superconductivity and critical current density**

   (a) The current in Sn when the surface field reaches the critical value is $I_{\max} = 2\pi r B/\mu_0 = 2\pi(5 \times 10^{-4})(0.2)/(4\pi \times 10^{-7}) = 500$ A. The corresponding current density is $j_{\max} = (500\text{ A})/(\pi(5 \times 10^{-4}\text{ m})^2) = 6.4 \times 10^8$ A/m^2.

   (b) Correspondingly for $B_{c2} = 24.5$ T in Nb_3Sn, $I_{\max} = 6.1 \times 10^4$ A and $j_{\max} = 7.810$ A/m^2.

   This maximum current density is close to the critical $j_c = 10^{11}$ A/m^2, but that is not always true. Remember that the $I_{\max}$ and $j_{\max}$ depend on the wire diameter, whereas j_c is a geometry-independent material property.

3. **Critical magnetic field of a superconductor**

   Consider a metal with some density of states per unit volume $g(E)$ in its normal state, which becomes a BCS superconductor below a critical temperature T_c. In BCS theory, the superconducting gap at zero temperature is given by $\Delta = 1.76 k_B T_c$.

   (a) If all the electrons with energy $E_F - \Delta < E < E_F$ pair up with binding energy Δ per pair, what is the gain in energy density of the superconductor relative to the normal metal?

   The number of electrons per unit volume in that energy range is approximately $g(E_F)\Delta$, so the number of pairs formed per unit volume is $g(E_F)\Delta/2$, each with energy gain Δ Therefore, the energy density gained relative to the normal metal is $g(E_F)\Delta^2/2$.

(b) What is the energy density incurred in expelling a magnetic field B due to the Meissner effect?

The supercodnuctor must produce an equal and opposite magnetic field B to cancel the external field. The energy density of this magnetic field is $B^2/(2\mu_0)$.

(c) Given that at the critical magnetic field B_c, it is no longer energetically favorable to expel the magnetic field, relate B_c to Δ (at $T = 0$).

The energy cost of expelling the magnetic field is equal to the energy gain of the superconducting state at $B = B_c$, so $B_c^2/(2\mu_0) = g(E_F)\Delta^2/2 \Rightarrow B_c = \sqrt{\mu_0 g(E_F)}\Delta$.

(d) Aluminum is face-centered cubic metal with a cubic lattice constant of 4.05 Å, which behaves almost perfectly like a free-electron metal with 3 free electrons per atom. What is its $g(E_F)$ in SI units ($\text{J}^{-1}\text{m}^{-3}$)?

The density of states in a free electron metal is

$$g(E) = \frac{\sqrt{E}}{2\pi^2}\left(\frac{\sqrt{2m}}{\hbar}\right)^3$$

and the Fermi energy is

$$E_F = \frac{\hbar^2}{2m}(3\pi^2 n)^{2/3}$$

where n is the number density of free electrons. Putting these together,

$$g(E_F) = \frac{m(3\pi^2 n)^{1/3}}{\pi^2\hbar^2}.$$

For aluminum, the unit cell volume is $\Omega = (4.05\ \text{Å})^3/4 = 1.66 \times 10^{-29}\ \text{m}^3$ and the number density is $n = 3/\Omega = 1.81 \times 10^{29}\ \text{m}^{-3}$. Substituting above, $g(E_F) = 1.43 \times 10^{47}\ \text{J}^{-1}\text{m}^{-3}$.

(e) Given that aluminum becomes a BCS superconductor below $T_c = 1.2$ K, estimate its zero-temperature critical magnetic field B_c in SI units (Tesla).

$$\begin{aligned} B_c &= \sqrt{\mu_0 g(E_F)}\Delta = \sqrt{\mu_0 g(E_F)} \cdot 1.76 k_B T_c \\ &= \sqrt{4\pi \times 10^{-7} \cdot 1.43 \times 10^{47}} \cdot 1.76 \cdot 1.38 \times 10^{-23} \cdot 1.2\ \text{T} \\ &\approx 0.012\ \text{T} \end{aligned}$$