HW1

MTLE-6120: Spring 2019

Due: Jan 21, 2019

1. **Resistivity of Cu-Au alloys**

   Read section 2.3.2 of Kasap and pay particular attention to Figure 2.12. The resistivity of alloys varies with atomic fraction X as $\rho = \rho_{\text{matrix}} + CX(1-X)$, where C is called the Nordheim coefficient. For Cu-Au alloys, for small Cu fractions in Au, $\rho_{\text{matrix}} \equiv \rho_{\text{Cu}}$, and for small Au fractions $\rho_{\text{matrix}} = \rho_{\text{Au}}$.

   (a) At 20° C, $\rho_{\text{Cu}} = 16.8$ nΩm and $\rho_{\text{Au}} = 24.4$ nΩm. For dilute alloys, $C_{\text{Au in Cu}} = 5500$ nΩm and $C_{\text{Cu in Au}} = 450$ nΩm. Plot the resistivity as a function of Au atom fraction in Cu, all the way from $X = 0$ (pure copper) to $X = 1$ (pure gold) using both the Cu and Au matrix formulae. Compare these two plots to the 'Quenched' line from Figure 2.12 in Kasap and explain.

   (b) Using the Cu in Au matrix formula, estimate the temperature coefficient of resistivity of the 50-50 (quenched) Cu-Au alloy and contrast it to that of the pure metals, $\alpha_{\text{Au}} = 0.0034$ K^{-1} and $\alpha_{\text{Cu}} = 0.0039$ K^{-1}.

   (c) From the Nordheim coefficients, calculate the cross-sections $\sigma_{\text{Au in Cu}}$ of electron scattering against Au impurities in Cu and $\sigma_{\text{Cu in Au}}$ of electron scattering against Cu impurities in Au. Assume that the electron velocity is approximately $v_F = 1.5 \times 10^6$ m/s in both cases. (Notation warning: σ is the standard symbol both for scattering cross section and conductivity.)

   (d) Hypothesize why $\sigma_{\text{Au in Cu}} \gg \sigma_{\text{Cu in Au}}$, and why the dilute Au in Cu limit is so different from other concentrations.

2. **Damped harmonic oscillator differential equation**

   This problem should help you revise / get comfortable solving ordinary differential equations and working with Fourier transforms.

   (a) Find the general solution of the homogeneous differential equation

   $$m\ddot{x}(t) - \gamma\dot{x}(t) + kx(t) = 0$$

   where $\dot{x}(t) \equiv \mathrm{d}x(t)/\mathrm{d}t$. Hint: try solutions of the form $x(t) = Ae^{iwt}$, find all w for which $A \neq 0$ (non-trivial solutions), and the general solution is the linear combination of all such solutions.

   (b) Find the specific solution of the previous differential equation that satisfies initial conditions $x(0) = x_0$ and $\dot{x}(0) = 0$.

   (c) Find the general solution of the forced harmonic oscillator

   $$m\ddot{x}(t) - \gamma\dot{x}(t) + kx(t) = Fe^{-i\omega t}.$$

   What is the 'steady-state' solution for large times $t \gg m/\gamma$? Hint: the general solution of an inhomogeneous equation is that of the corresponding homogeneous equation + a particular solution $x_p(t)$; in this case, try $x_p(t) = x_1 e^{-i\omega t}$.

   (d) Show that the steady-state solution above can be easily 'solved in the Fourier domain' by assuming $x(t) = xe^{-i\omega t}$ and replacing $\mathrm{d}/\mathrm{dt} \to -i\omega$ in the differential equation.

3. **Optical response of bound electrons**

In class, we solved the equation of motion of a free electron to derive Drude theory. Now consider the opposite limit, where electrons are tightly bound to atoms, which is the most reasonable classical description for electrons in insulating materials. Specifically, assume that each electron is attached to an atom with spring constant k and damping constant γ, that is if it is displaced by $\vec{r}(t)$, there will be a restoring force $-k\vec{r}(t)$ and damping force $-\gamma\frac{\mathrm{d}\vec{r}}{\mathrm{d}t}$.

(a) Solve the equation of motion for an electrons position $\vec{r}(t)$, when an oscillatory electric field $\vec{E}(t) = \vec{E}e^{-i\omega t}$ is applied. What is the induced dipole moment $\vec{p}$ due to this electron? (Hint: this is extremely easy in the frequency domain, as set up by the previous problem.)

(b) Assume that there is a number density n_b of such bound electrons that all respond the same way as above. Calculate the polarization $\vec{P}$, displacement field $\vec{D}$, and hence the dielectric function $\epsilon(\omega)$.

(c) Express the dielectric function in terms of the spring resonant frequency $\omega_0 = \sqrt{k/m}$ and damping rate $\Gamma = \gamma/m$, and the plasma frequency of bound electrons $\omega_{pb} = \sqrt{n_b e^2/(m\epsilon_0)}$. How does this differ from the Drude result for free electrons?

(d) In a real metal, some electrons are free and the rest are bound to the atoms. In copper, for example, 1 s electron per atom is effectively free, while 10 d electrons are bound tightly to the atoms. (The remaining 'core' electrons are bound so tightly that we can assume exactly fixed, and ignore them.) Write the dielectric function for this general case with n free electrons and n_b spring-bound electrons.