HW6 solution

MTLE-6120: Spring 2017

Due: Apr 17, 2017

1. **Critical magnetic field of a superconductor**

   Consider a metal with some density of states per unit volume $g(E)$ in its normal state, which becomes a BCS superconductor below a critical temperature T_c. In BCS theory, the superconducting gap at zero temperature is given by $\Delta = 1.76 k_B T_c$.

   (a) If all the electrons with energy $E_F - \Delta < E < E_F$ pair up with binding energy Δ per pair, what is the gain in energy density of the superconductor relative to the normal metal?

   The number of electrons per unit volume in that energy range is approximately $g(E_F)\Delta$, so the number of pairs formed per unit volume is $g(E_F)\Delta/2$, each with energy gain Δ Therefore, the energy density gained relative to the normal metal is $g(E_F)\Delta^2/2$.

   (b) What is the energy density incurred in expelling a magnetic field B due to the Meissner effect?

   The supercodnuctor must produce an equal and opposite magnetic field B to cancel the external field. The energy density of this magnetic field is $B^2/(2\mu_0)$.

   (c) Given that at the critical magnetic field B_c, it is no longer energetically favorable to expel the magnetic field, relate B_c to Δ (at $T = 0$).

   The energy cost of expelling the magnetic field is equal to the energy gain of the superconducting state at $B = B_c$, so $B_c^2/(2\mu_0) = g(E_F)\Delta^2/2 \Rightarrow B_c = \sqrt{\mu_0 g(E_F)}\Delta$.

   (d) Aluminum is face-centered cubic metal with a cubic lattice constant of 4.05 Å, which behaves almost perfectly like a free-electron metal with 3 free electrons per atom. What is its $g(E_F)$ in SI units (J^{-1}m^{-3})?

   The density of states in a free electron metal is

$$g(E) = \frac{\sqrt{E}}{2\pi^2}\left(\frac{\sqrt{2m}}{\hbar}\right)^3$$

   and the Fermi energy is

$$E_F = \frac{\hbar^2}{2m}(3\pi^2 n)^{2/3}$$

   where n is the number density of free electrons. Putting these together,

$$g(E_F) = \frac{m(3\pi^2 n)^{1/3}}{\pi^2\hbar^2}.$$

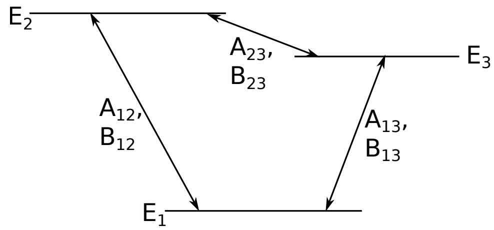
   For aluminum, the unit cell volume is $\Omega = (4.05 \text{ Å})^3/4 = 1.66 \times 10^{-29}$ m^3 and the number density is $n = 3/\Omega = 1.81 \times 10^{29}$ m^{-3}. Substituting above, $g(E_F) = 1.43 \times 10^{47}$ J^{-1}m^{-3}.

   (e) Given that aluminum becomes a BCS superconductor below $T_c = 1.2$ K, estimate its zero-temperature critical magnetic field B_c in SI units (Tesla).

$$\begin{aligned} B_c &= \sqrt{\mu_0 g(E_F)}\Delta = \sqrt{\mu_0 g(E_F)} \cdot 1.76 k_B T_c \\ &= \sqrt{4\pi \times 10^{-7} \cdot 1.43 \times 10^{47}} \cdot 1.76 \cdot 1.38 \times 10^{-23} \cdot 1.2 \text{ T} \\ &\approx 0.012 \text{ T} \end{aligned}$$

2. **Optical pumping and fluorescence**

Cosnider the minimal three-level system necessary for fluorescence (and lasing) as shown below. A pump light of intensity I_{pump} is tuned to a frequency matching $E_2 - E_1$, and assume the intensities at other frequencies are small enough that stimulated emission is negligible for the other transitions. Assume A and B coefficients for each pair of states as shown.



(a) Write the differential equations governing the kinetics of N_1, N_2 and N_3: the populations (electrons / volume) for the three states.

For each pair of states, there is a rate $AN_{\text{upper}} + BIN_{\text{upper}}$ going from the upper to lower state due to spontaneous and stimulated emission, and a rate BIN_{lower} going form the lower to the upper state due to absorption. This results in

$$\begin{aligned}\dot{N}_1 &= [A_{12}N_2 + B_{12}I_{\text{pump}}(N_2 - N_1)] + A_{13}N_3 \\ \dot{N}_2 &= -[A_{12}N_2 + B_{12}I_{\text{pump}}(N_2 - N_1)] - A_{23}N_2 \\ \dot{N}_3 &= A_{23}N_2 - A_{13}N_3\end{aligned}$$

(b) In steady state, find the ratio N_3/N_1 in order to determine the condiction for population inversion ($N_3 > N_1$). Find and interpret the $I_{\text{pump}} \to \infty$ limit of this criterion.

In steady state, $\dot{N}_1 = \dot{N}_2 = \dot{N}_3 = 0$, so that

$$\begin{aligned}0 &= [A_{12}N_2 + B_{12}I_{\text{pump}}(N_2 - N_1)] + A_{13}N_3 \\ 0 &= -[A_{12}N_2 + B_{12}I_{\text{pump}}(N_2 - N_1)] - A_{23}N_2 \\ 0 &= A_{23}N_2 - A_{13}N_3\end{aligned}$$

Since we want to compare N_3 and N_1, use the last equation to replace N_2 in favor of N_3 in the second equation (same result if done in first equation). This gives the population inversion criterion:

$$\frac{N_3}{N_1} = \frac{B_{12}I_{\text{pump}}}{A_{13}\left(1 + \frac{A_{12}+B_{12}I_{\text{pump}}}{A_{23}}\right)} > 1$$

In the limit of $I_{\text{pump}} \to \infty$:

$$\frac{N_3}{N_1} = \frac{A_{23}}{A_{13}} > 1$$

which simply means that the spontaneous emission rate from $3 \to 1$ must be slower than that from $2 \to 3$ (which feeds state 3). Therefore, having a forbidden transition (selection rule) is useful to slow down $3 \to 1$.

(c) What is the net power density (rate of energy change per unit volume) absorbed from the pump light into the electrons? Assume that stimulated emission puts energy back into I_{pump} (best case scenario for efficiency), while the energy from spontaneous emission is lost. Just write the answer in terms of instantaneous N_1, N_2, N_3 (don't solve for the Ns).

The net rate per unit volume at which photons are being taken from the pump (absorption - stimulated emission) is $B_{12}I_{\text{pump}}(N_1 - N_2)$. These photons have energy $(E_2 - E_1)$, so the net absorbed power density is $B_{12}I_{\text{pump}}(N_1 - N_2)(E_2 - E_1)$.

(d) Similarly, what is the power density output from the $3 \to 1$ transition (fluorescence)? Again, just express in terms of N_1, N_2, N_3 as needed.

The rate of the fluorescence transition is $A_{13}N_3$ and the photon energy is $E_3 - E_1$, so that the power density emitted is $A_{13}N_3(E_3 - E_1)$.

(e) What is the energy efficiency of the fluorescence process in steady state, and how does it depend on I_{pump}? (This time, solve for the Ns in terms of the A, B parameters and interpret!)

The energy efficiency is the power density emitted / power density absorbed

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\begin{aligned}
\text{Efficiency} &= \frac{A_{13}N_3(E_3 - E_1)}{B_{12}I_{\text{pump}}(N_1 - N_2)(E_2 - E_1)} \\
&= \frac{A_{13}N_3(E_3 - E_1)}{B_{12}I_{\text{pump}}\left(N_1 - N_3\frac{A_{13}}{A_{23}}\right)(E_2 - E_1)} && \text{last steady state equation} \\
&= \frac{A_{13}\frac{N_3}{N_1}(E_3 - E_1)}{B_{12}I_{\text{pump}}\left(1 - \frac{N_3}{N_1}\cdot\frac{A_{13}}{A_{23}}\right)(E_2 - E_1)} \\
&= \frac{A_{13}\frac{B_{12}I_{\text{pump}}}{A_{13}\left(1+\frac{A_{12}+B_{12}I_{\text{pump}}}{A_{23}}\right)}(E_3 - E_1)}{B_{12}I_{\text{pump}}\left(1 - \frac{B_{12}I_{\text{pump}}}{A_{13}\left(1+\frac{A_{12}+B_{12}I_{\text{pump}}}{A_{23}}\right)}\cdot\frac{A_{13}}{A_{23}}\right)(E_2 - E_1)} && \text{using part (b) solution} \\
&= \frac{A_{23}}{A_{23} + A_{12}}\cdot\frac{E_3 - E_1}{E_2 - E_1}
\end{aligned}
$$

The result is independent of the pump power and depends only on the ratio of two A parameters, and the energies. The first factor expresses the fraction of electrons excited to state 2 that end up in state 3 instead of going back to state 1 via spontaneous emission. The second factor accounts for the fact that the energy of the emitted photon is smaller than the absorbed one.